Royal Meteorological Society

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CANADIAN BRANCH

25¢

VOL. 3 NO. 5

Published By

ROYAL METEOROLOGICAL SOCIETY, CANADIAN BRANCH

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Presented at the regular monthly meeting of the Royal Meteorological Society, Canadian Branch, Toronto, Thursday, May 29, 1952.

ERRATA

Page 4, Equation 9, should read:

$$k_\nu = \frac{S\alpha}{\pi} \frac{1}{(\nu-\nu_0)^2 + \alpha^2}, \qquad (9)$$

Page 5, Equation 10, should read:

$$\tau_I = \int e^{-\frac{\alpha m}{\pi} \sum_n \frac{S_n}{(\nu-\nu_n)^2+\alpha^2}} d\nu \Big/ \int d\nu. \qquad (10)$$

Page 5, line 17, the expression for the generalized absorption coefficient, should read:

$$\left(2\pi S\alpha / d^2\right)$$

Page 10, line 4 from the bottom, should read "the radiation emitted by layer one is, since $T_b = T_1$,"

Page 12, Equation 20, the second integral should read:

$$\int_{T_0}^{T'} \frac{\partial F_b \phi}{\partial T} dT$$

THE DETERMINATION OF RADIATIVE FLUXES

DUE TO ATMOSPHERIC WATER VAPOUR

SUMMARY

The formal problem of radiative flux determination is developed, and it is indicated why the integration over all wave lengths should precede the introduction of actual atmospheric data into flux computation. A technique to do this is outlined, commencing with the evaluation of intensity transmission for incident black body radiation, its conversion to flux transmission, and the integration over the entire spectrum. The properties of suitable intensity transmission functions are described. The effects of pressure and temperature on intensity transmission and integrated flux transmission are outlined. A procedure for obtaining the transmission of black body radiation by a number of layers of varying pressure and temperature is next presented, and applied to the basic problem of determination of total radiative fluxes. The relation of this technique to those advocated by Elsasser (1942), Robinson (1947), and Kaplan (1952) is presented, together with brief derivations of the relations utilized by these authors.

INTRODUCTION

Classical studies of the general circulation were based on the energy budget of the earth and its atmosphere, involving radiative fluxes of solar and atmospheric (or terrestrial) radiation at the surface and at the outer limit of the atmosphere.

In recent years, largely as a result of the influence of the U. of Chicago School of Thought, emphasis shifted to the purely dynamic aspects of the general circulation. The energy input driving this circulation cannot be neglected, however, and at present there are indications of a swing of the pendulum back to energy considerations, largely as a result of the M.I.T. studies of meridional fluxes of sensible and latent heat and of angular momentum. Classical estimates of radiative fluxes are now no longer considered sufficiently accurate for the purpose, so that there has been a revived interest in radiation studies. Measurements of solar radiation are now very numerous; this is not true of long wave radiation, however, so that it is necessary to derive the desired fluxes from computations based largely on theory.

THE FORMAL PROBLEM OF RADIATIVE FLUX DETERMINATION

Consider a beam of radiation, of specific intensity I_λ, incident on a thin layer, with vertical absorbing mass dm, at an angle θ to the vertical (θ = zenith angle).

From Beer's Law, the decrease in intensity due to absorption is given by:

$$-dI_\lambda = I_\lambda \cdot k_\lambda \, dm \sec\theta , \qquad (1)$$

the effective absorbing mass for a slant path being dm sec θ, and where k_λ is the absorption coefficient, a function of wave length (λ), pressure (p), temperature (T), and the nature of the absorbent. If there are n various absorbents the fractional absorption is

$$-\frac{dI_\lambda}{I_\lambda} = \sec\theta \sum_n \left(k_\lambda\, dm\right)_n . \qquad (2)$$

The radiative beam leaving dm is augmented by the emission of the layer. From Kirchhoff's Law, the emitted intensity is

$$dI_\lambda = E_\lambda \cdot \sec\theta \sum_n \left(k_\lambda\, dm\right)_n , \qquad (3)$$

where E_λ represents the specific black-body intensity, given by Planck's Law as a function of λ and T only.

In order to evaluate the total radiative flux of energy crossing a specific reference level (m = 0), it is sufficient to consider the amount of the energy emitted by dm which is transmitted through m, and sum this quantity for all emitting layers to the top of the atmosphere. From (2), the fractional transmission is obtained as:

$$e^{-\sec\theta \sum_n \int_0^m k_\lambda\, dm} .$$

Thus the energy emitted by dm, in the direction θ, and in the wave length range from λ to $\lambda + d\lambda$, and transmitted to the level m=0, is

$$dI_\lambda = E_\lambda \sec\theta \sum_n \left(k_\lambda dm\right) e^{-\sec\theta \sum_n \int_0^m k_\lambda dm} \qquad (4)$$

where the summation is over all possible absorbers - water vapour, carbon dioxide, methane, nitrous oxide, ozone, and carbon monoxide.

The above expression, (4), must now be integrated over all zenith angles to convert to a flux (hemispheric radiation) using the relation

$$df_\lambda = 2\pi \int_0^{\pi/2} dI_\lambda \cos\theta \sin\theta \, d\theta \qquad (5)$$

This must now be integrated over all wave lengths from zero to infinity and finally integrated in the vertical from the level m=0 to the level of maximum absorbing masses (m^1, say). Using (4) and (5), performing these operations, and simplifying, there will be obtained the following relation for the total radiative flux crossing the level m = 0 from above:

$$F = -2 \int_0^\infty d\lambda \int_0^{m'} f_b \, d\left\{ E_{i_3}\left(\sum_n \int_0^m k_\lambda\, dm \right) \right\} \qquad (6)$$

where: $Ei_3(x) = \int_1^\infty e^{-xy}\,dy/y^3$ (a tabulated function),

and f_b represents the specific black body flux (πE_λ) at a level specified by m.

It is clear that the integral (6) will have to be evaluated by numerical means and that the final integration will be over λ. Now, k_λ for the various absorbers varies extremely rapidly and irregularly with λ, so that the integral from $m = 0$ to $m = m^1$ would have to be evaluated for some 10,000 wave lengths in order to ensure accuracy for the final integration.

An alternative approach is to integrate (6) over a very narrow range of wave lengths, for which f_b is virtually constant, and later perform a summation over say 100 spectral intervals. In this case (6) becomes

$$F = -2\sum_{\Delta\lambda}\int_0^{m'} f_b\, d\left\{\int_\lambda^{\lambda+\Delta\lambda} Ei_3\left(\Sigma_n \int_0^m k_\lambda\, dm\right) d\lambda\right\} \qquad (6a)$$

Even this method would not be feasible for normal or routine use, so that there is a definite need for a different technique, in which wave length variations are taken into account in advance, and only an integration or summation in the vertical through the atmosphere is required.

A SIMPLIFIED APPROACH TO RADIATIVE FLUX DETERMINATION

It will be shown that the radiative flux crossing any level can be computed if we know the flux transmission (i.e., for radiation over an entire hemisphere), integrated over the entire spectrum, by a layer of known absorbing mass, pressure, and temperature, the incident radiation being black-body radiation corresponding to a fixed radiating temperature, T_b. For the present it will be assumed that only a single absorber, water vapour, is present.

Let τ_I represent the fraction of black body radiative intensity transmitted at zenith angle θ, over a narrow spectral range for which E_λ varies only slightly. Let τ_f represent the fraction of the increment of black body flux in that spectral range, ΔF_b, transmitted by the layer; and let ϕ represent the fraction of the total black body flux, F_b, integrated over the entire spectrum, transmitted by the layer.

Analogous to (5): $$\tau_f = 2\int_1^\infty \tau_I(mx)\,dx/x^3 \qquad (7)$$

Moreover: $$F_b\,\phi = \Sigma\,\tau_f\,\Delta F_b \qquad (8)$$

The operations indicated by (7) and (8) can be performed once and for all, since they do not depend on any particular atmospheric stratification, but only on fixed values of m, p, T and T_b. There remains the problem of evaluating the intensity transmission, τ_I , over a narrow spectral range, as a function of absorbing mass, pressure, and temperature in terms of parameters of the specific spectral range being investigated.

INTENSITY TRANSMISSION FOR A BAND SPECTRUM

The Quantum Theory of Radiation tells us that radiation is absorbed when a molecule (or atom) changes its energy state from one discrete value to a higher discrete value; for emission, the reverse is true. The frequency of the radiation concerned, ν , is related to the energy change, $E_2 - E_1$, by the relation:

$$E_2 - E_1 = h\,\nu ,$$

where h is Planck's constant. Since E can have only a series of specific discrete values, radiation will be emitted or absorbed only for discrete frequencies. Each permissible energy change is thus proportional to a frequency, at which radiation may be emitted or absorbed. The frequency is related to the wave length by the relation: $\nu = c/\lambda$, where c represents the velocity of light. In radiation studies it is convenient to define frequency as $1/\lambda$, the quantity usually referred to as the wave number elsewhere, and this definition will be used here.

As a result of collisions between radiating molecules and molecules of the same or other species, emission and absorption may take place at frequencies slightly different from the theoretical frequencies. Thus a spectroscopic examination of emitted radiation reveals the presence of radiative energy over very narrow spectral intervals, known as spectral lines, the lines themselves being distinct and separate when viewed under high resolution (or magnification). These spectral lines are not distributed at random over the entire spectrum but are closely grouped together in spectral bands.

From the Lorentz theory of collision broadening of spectral lines it is possible to obtain an expression for the absorption coefficient as a function of frequency (i.e. $1/\lambda$) due to a single spectral line. This expression is:

$$k_\nu = \frac{S\alpha}{\pi}\,\frac{1}{(\nu + \nu_0)^2 + \alpha^2} , \qquad (9)$$

where S is the total intensity of the line, a temperature-dependent function related to the probability of the energy change producing the line, ν_0 is the central frequency of the line, and α is the half-width of the line, so-called since k_ν at $\nu_0 \pm \alpha$ is one-half its value at ν_0.

The total absorption coefficient, at frequency ν, will be the sum of expressions of the form of (9), one for each line in the spectrum. In general, over a restricted interval of the water vapour spectrum, lines are rather irregularly spaced and their intensities also vary greatly from line to line. The half-width, α, is sensibly constant, however, and depends only on the total pressure. Making use of Beer's Law, it can be shown that the mean intensity transmission over a narrow spectral interval is:

$$\tau_I = \int e^{-\frac{\alpha m}{\pi} \Sigma_n \frac{S_n}{(\nu-\nu_m)^2 + \alpha^2}} d\nu \Big/ \int d\nu . \qquad (10)$$

It is not possible to integrate the above equation in the general case of n lines with a random distribution of frequency and intensity. The problem does become manageable, however, if it is assumed that the actual spectrum would produce the same effect as a schematic spectrum containing equi-distant lines of equal intensity. This case was solved by Elsasser (1942), who obtained

$$\tau_I = \sinh\beta \int_0^{\infty} e^{-y\cosh\beta} J_0(iy)\, dy , \qquad (11)$$

where $y = Sm/d \sinh\beta = l_m^y/\beta\sinh\beta$, $\beta = 2\pi\alpha/d$, $S =$ mean line intensity, $d =$ mean line spacing, l is Elsasser's generalized absorption coefficient ($2\pi S\alpha/d^2$), and $J_0(iy)$ is the Bessel function of zero order with an imaginary argument (a tabulated function). From data of quantum mechanics, S and d can be evaluated for a series of spectral intervals, containing say 10-20 lines ($\Delta\nu = 20\, cm^{-1}$), and α can be obtained from spectroscopic determinations. Eqn(11) can be integrated numerically as a function of the two non-dimensional parameters - lm and β, and hence tabulated in advance of computations with any specific absorbent for any specific spectral range.

According to the Lorentz Theory, α varies linearly with the total pressure. S will be a complex function of temperature only, and d will be independent of both pressure and temperature. If l_o and β_o represent values at a standard pressure, p_o, it follows that:

$$l = l_o\, p/p_o , \quad \beta = \beta_o\, p/p_o . \qquad (12)$$

PROPERTIES OF INTENSITY TRANSMISSION FUNCTIONS

The intensity transmission function (11) can be approximated by simple functions in two limiting cases. For very thin layers of absorbent (lm small), especially for lines close together (β large):

$$\tau_I \simeq e^{-lm/\beta} \simeq 1 - lm/\beta . \qquad (13)$$

Thus for very small absorbing masses, the absorption $(1-\tau_I)$ is approximately proportional to the absorbing mass, acting in the same manner as the simple Beer's Law exponential function.

For very thick layers of absorbent (lm large), especially for lines far apart (β small):

$$\tau_I = 1 - \Phi\left(\ell m/2\right)^{1/2}, \tag{14}$$

where $\Phi(x)$, the error function is: $\Phi(x) = 2\pi^{-\frac{1}{2}}\int_0^x e^{-x^2}dx.$

It may be noted that Elsasser (1942) used (14) instead of (11) on his Radiation Chart, with a semi-empirical correction for very small lm to obtain better agreement with (13).

For small values of (lm), (14) becomes, approximately

$$\tau_I \simeq 1-(2\,\ell m/\pi)^{1/2}. \tag{15}$$

This predicts, for thin layers, an absorption varying with the square root of the absorbing mass. This behaviour would only be experienced to a very limited degree in practice, since the error-function relation itself is only valid for moderately thin layers if β is very small. In those portions of the water vapour spectrum with lines far apart, the lines themselves are generally exceedingly weak, so that the major contribution to the absorption coefficient at a given frequency comes from distant strong lines.

At extreme band wings, or between bands, the above conditions apply, and the absorption coefficient varies slowly with frequency. This gives essentially a continuous, but weak, absorption, with an effective absorption coefficient over a narrow spectral range given from (9) as:

$$k = (\alpha/\pi)\,\Sigma_n S_n/(\nu-\nu_n)^2 = k_0\, p/p_0 \;; \tag{16}$$

since α is of the order of 0.1 cm^{-1} and $(\nu-\nu_n)$ for the stronger lines is of the order of 100 cm^{-1}. The transmission, from Beer's Law, will be

$$\tau_I = e^{-km} \tag{17}$$

Equations (11) and (17) can now be used to obtain τ_f (using (7)), and ϕ (using (8)). The resulting integrated flux transmission ϕ, applicable to the entire spectrum, will depend on the absorbing mass, pressure, and temperature of a layer, as well as on the radiating temperature of the black-body radiation incident on the layer. This latter dependancy arises because a warmer radiating black-body emits a greater percentage of its radiation at the shorter wave lengths. Values of ϕ, obtained in the above manner, have been tabulated by Godson (1952).

EFFECT OF PRESSURE UPON TRANSMISSION OF RADIATION

It would be very convenient if techniques could be developed to reduce the number of variables involved in the determination of radiative fluxes. One possibility is to adjust the absorbing mass in a layer to a value which, at a standard pressure and temperature, would give the same transmission as the actual layer. In the past the temperature effect on absorption has been generally neglected, and the pressure effect taken care of in the above manner. In terms of the integrated flux transmission this is equivalent to setting:

$$\phi(m, p, T; T_b) \equiv \phi(m_o, p_o, T; T_b),$$

and assuming that (m_o/m) is a function only of p, but not of ϕ or T.

In order to investigate the pressure effect on ϕ, it will be advantageous to consider its effect on τ_I, and to assume that an appropriate form for (m_o/m) is $(p/p_o)^n$, where n is a pressure-correction exponent. For thin absorbing layers, especially for lines close together, from (12) and (13),

$$\tau_I \simeq e^{-lm/\beta} = e^{-l_o m/\beta_o} \simeq e^{-l_o m_o/\beta_o}.$$

Thus $m = m_o$ and n is zero, absorption being independent of pressure. When the lines are sufficiently broad (relative to their displacement) to give a continuous type of absorption, further increases of pressure have no effect, a well-known result in spectroscopy.

For deep absorbing layers, especially for lines far apart, from (12) and (14),

$$\tau_I \simeq 1 - \Phi(lm/2)^{1/2} \simeq 1 - \Phi(l_o m p/2p_o)^{1/2} \simeq 1 - \Phi(l_o m_o/2)^{1/2}.$$

In this case, $m_o = mp/p_o$ and n is unity, indicating a linear pressure correction to absorbing mass.

For any absorbing layer, in spectral regions at extreme band wings or between bands, from (16) and (17),

$$\tau_I = e^{-km} = e^{-k_o m p/p_o} = e^{-k_o m_o}.$$

In this case as well, $m_o = mp/p_o$ and n is again unity.

Near band centres, the appropriate pressure correction varies with transmission, pressure and the ratio of half-width to line spacing (i.e., with β_o). The pressure-correction exponent, n varies from zero (very thin layers) to unity (very thick layers). For the same τ_I and β_o, n increases as p decreases; for the same τ_I and p, n increases rapidly as β_o decreases.

Some ten years ago, it was considered that a square-root pressure correction was generally valid for all spectral regions, all pressures, and all absorbing masses, i.e., that $m_o = m\ (p/p_o)^{1/2}$. This was based chiefly on the work of Elsasser (1942), and has been followed by many authors since that time. Recently, however, it has been realized that a square-root pressure correction is approximately true only for a very limited range of transmissions or absorbing masses; this has been clearly demonstrated by Cowling (1950) and Kaplan (1952). Elsasser based his conclusions on laboratory experiments with very short path lengths. Even when such measurements apply to the entire spectrum, the short path lengths ensure that appreciable absorption takes place only near band centres, a region where n increases gradually from zero to unity, with an average value of the order of 0.5 for the path lengths studied. For the much deeper layers of the atmosphere, especially when spectral regions distant from band centres are considered, the appropriate value of n is very close to unity.

This can be illustrated by a study of the transmission of 300°K black-body radiation by atmospheric layers at 300°K and pressures of 1.0 and 0.5 atmospheres. The thickness of the layers concerned can be visualized by assuming a mixing ratio of 9.8 gm/kgm (corresponding to a dew point near 60°F), and evaluating the pressure thickness of the equivalent layer (m_o, gm/cm^2) in mb. A thickness of 1 mb corresponds roughly to a depth of 10 m or 30 ft. The value of n is strictly valid only at 0.5 atmospheres, but does not vary greatly with pressure.

TABLE I . PRESSURE-CORRECTION EXPONENT, AS A FUNCTION OF m_o

ϕ (%)	96.8	94.9	91.3	87.8	78.0	69.5	58.2
m_o(gm/cm^2)	0.0001	0.0002	0.0005	0.001	0.005	0.02	0.1
Δp (mb)	0.01	0.02	0.05	0.1	0.5	2.0	10.0
n	0.425	0.59	0.735	0.80	0.875	0.945	0.985

It can be seen that a square-root pressure correction (n = 0.5) is only appropriate to the very thin layers customarily studied in the laboratory. For atmospheric layers exceeding a few mb in depth, a linear pressure correction is highly accurate.

EFFECT OF TEMPERATURE UPON TRANSMISSION OF RADIATION.

TABLE II. INTEGRATED FLUX TRANSMISSION, ϕ (expressed as a percentage), at 1 atm pressure.

m (gm/cm^2) T_b(°K):	220		300	
T (°K):	220	300	220	300
0.0001	95.9	95.3	97.1	96.8
0.001	84.9	82.6	89.3	87.8
0.01	69.0	64.8	76.7	73.7
0.1	51.6	47.6	61.5	58.2
1.0	22.2	20.1	30.5	28.5
10.0	0.5	0.5	1.3	1.3

As the temperature of the absorbent increases, the population of the higher energy levels increases relative to that of the lower energy levels. Thus the intensities of the weakest lines increase with temperature while those of the strongest lines decrease. Absorption thus increases with temperature at band wings, decreases slightly at band centres, and is approximately constant between bands. As the above table shows, these effects produce little change in transmission for very thin layers (band centre absorption only) or for very thick layers (incomplete absorption between bands only), but produce a marked decrease in transmission for intermediate ranges of transmission as the absorbent temperature increases, the black-body radiating temperature remaining constant.

The effect of increase of black-body radiating temperature on the integrated flux transmission is in the opposite sense to that of increase of absorbent temperature, and is numerically more significant over the entire transmission range. As the radiating temperature increases, the wave length of maximum intensity decreases, and a greater percentage of the energy is distributed over regions of weak absorption. This results in an increased transmission, as can be seen from the above table.

MULTIPLE-LAYER TRANSMISSION

If absorbing masses of thin layers could be individually corrected for pressure and temperature, all layers from the reference level to an emitting layer could be considered as a single unit for the computation of the transmission of the emitted radiation, with a corrected absorbing mass equal to the sum of the individual corrected absorbing masses. Unique corrections to a standard pressure and temperature are not possible, however. For example, a very thin layer adjacent to the emitting layer should receive no pressure correction, while a layer near the reference level should receive a linear pressure correction. It is still possible, however, to adjust the absorbing mass in one layer to the conditions in an adjacent layer, and then amalgamate the two layers into a single layer.

Consider the simple case of transmission of black-body radiation, emitted at T_b, through two layers — m_1 at p_1, and T_1, and m_2 at p_2 and T_2. It is possible to find an absorbing mass m_2' at p_2 and T_2 which would have the same transmission as m_1 at p_1 and T_1, i.e.,

$$\phi(m_1, p_1, T_1; T_b) = \phi(m_2', p_2, T_2; T_b)$$

The net transmission should be nearly the same as for $(m_2 + m_2^1)$ at p_2 and T_2, or for $(m_1 + m_1^1)$ at p_1 and T_1, where m_1^1 is defined by

$$\phi(m_2, p_2, T_2; T_b) = \phi(m_1', p_1, T_1; T_b)$$

A detailed analysis of the problem shows that the net transmission is given, with sufficient accuracy, by the average of these two empirically determined net transmissions. Hence

$$2\phi = \phi(m_1 + m_1', p_1, T_1; T_b) + \phi(m_2 + m_2', p_2, T_2; T_b).$$

This technique can be extended tocover any number of layers, by proceeding through the assembly of layers in two directions.

DETERMINATION OF TOTAL RADIATIVE FLUX ACROSS ANY LEVEL

Consider the net downward radiative flux at some level in the atmosphere above which there may be considered to be three layers, numbered consecutively downwards. The net downward flux at the reference level is thus the sum of the radiation emitted by layer one and transmitted through layers two and three, plus the radiation emitted by layer two and transmitted through layer three, plus the radiation emitted by layer three.

From Kirchhoff's Law and the Stefan-Boltzmann Law ($F_b = \sigma T_b^4$), the radiation emitted by layer one , since $T_b = T_1$,

$$\sigma T_1^4 [1 - \phi(m_1, p_1, T_1; T_1)].$$

The fraction of this radiation transmitted through layers two and three cannot be determined directly since the integrated flux transmission functions apply only to incident black-body radiation.

However, the two quantities in the above expression can be transmitted separately, since σT_1^4 is black-body radiation and the second term represents the amount of black-body radiation transmitted through layer 1, which may be written as $\sigma T_1^4 \phi(1;T_1)$. Thus the radiation from layer one reaching the reference level is:

$$\sigma T_1^4 \left[\phi(2+3;T_1) - \phi(1+2+3;T_1)\right].$$

The multiple-layer transmissions in the above expression will be found by the method of the last section. In a similar manner, the radiation emitted by layer two, and transmitted through layer three, is:

$$\sigma T_2^4 \left[\phi(3;T_2) - \phi(2+3;T_2)\right].$$

Using the above technique, the total radiative flux crossing any reference level from above or below can be determined, provided that a large number of integrated flux transmission graphs are available for a series of values of p, T and T_b.

SIMPLIFIED GRAPHICAL TECHNIQUES IN USE FOR RADIATIVE FLUX

If the emitting layer is considered as infinitesimally thin, the element of flux emitted and subsequently transmitted to the reference level will be

$$dF = F_b\left[\phi(m;T_b) - \phi(m+dm;T_b)\right]$$
$$\text{or,} \quad dF = -F_b\left(\frac{\partial\phi(m,p,T;T_b)}{\partial m}\right)_{T_b} dm. \qquad (18)$$

The multi-layer transmission, ϕ, cannot be expressed in terms of the variation of m, p, and T through the layer, so that the above derivative cannot be evaluated analytically.

For a graphical solution of (18), it is necessary to reduce the four independent variables to two. This Elsasser (1942) did by neglecting the effect of absorbent temperature on transmission and by adopting a square-root pressure correction to the absorbing mass. Actually, a linear pressure correction would have been more appropriate, on the average. Kaplan (1952) suggested that, if Elsasser's Radiation Chart is to be used, the absorbing mass is best adjusted by multiplication by 0.4 (p/1000), the factor 0.4 arising since Elsasser used too large a value for the half-width. Adopting a linear pressure correction, the pressure error is less significant than the temperature error arising from the neglect of the effect of absorbent temperature. This is illustrated by the integrated flux transmission data of the following table.

TABLE III. INTEGRATED FLUX TRANSMISSION, ϕ, EXPRESSED AS A PERCENTAGE.

mp/p_o (gm/cm^2)	T_b (°K):	300	300	300	300	300	300	300
	T (°K):	300	300	260	260	260	220	220
	p (atm):	1.0	0.5	1.0	0.5	0.1	0.5	0.1
0.001		87.8	87.1	88.8	88.0	87.6	88.6	88.2
0.01		73.7	73.3	75.3	74.8	74.7	76.2	76.0
0.1		58.2	58.1	60.0	59.9	59.9	61.4	61.3
0.5		39.7	39.7	41.8	41.7	41.7	42.4	42.3
2.0		16.5	16.5	18.1	18.1	18.1	17.8	17.8

With Elsasser's assumptions, temperature only enters as the black-body temperature, T_b, determining both F_b and $\phi(m, T_b)$; it will henceforth be denoted simply by T. The total flux is thus

$$F = -\int_0^{m'} F_b \frac{\partial \phi}{\partial m} dm, \tag{19}$$

where m' represents the total absorbing mass above the reference level corresponding to a temperature T' in the stratosphere, and where $m = 0$ at the reference level with temperature T_o. Since ϕ is a function of m and T, (19) can be written as:

$$F = -\int_0^{m'} F_b \frac{d\phi}{dm} dm + \int_0^{m'} F_b \frac{\partial \phi}{\partial T} \frac{dT}{dm} dm$$

Integrating the first integral by parts, there is obtained:

$$F = -\Big[F_b \phi\Big]_0^{m'} + \int_{T_o}^{T'} \phi \frac{dF_b}{dT} dT + \int_{T_o}^{T'} F_b \frac{\partial \phi}{\partial T} dT$$

Since F_b is a direct function only of T, the last two integrals can be combined, and the total radiative flux can be written as:

$$F = \int_0^{T_o} \left(\frac{\partial F_b \phi}{\partial T}\right)_{m=0} dT + \int_{T_o}^{T'} \frac{\partial F_b \phi}{\partial T} dT + \int_{T'}^{0} \left(\frac{\partial F_b \phi}{\partial T}\right)_{m=m'} dT \tag{20}$$

Since the first and last integrals are to be evaluated at a constant m, and since $F_b = 0$ for $T = 0$, they have the values $F_b(T_o)\,\phi(0, T_o)$ and $-F_b(T')\,\phi(m', T')$, respectively. The second integral is to be evaluated at constant m, but this m value varies with T from 0 at To to m' at T'. Eqn (20) is the relation on which Elsasser based his radiation chart, his function $Q(m, T)$ being simply $\partial F_b \phi / \partial T$, although he obtained values of Q without directly evaluating ϕ first. The flux in (20) is clearly an area on a diagram with coordinates T and $\partial F_b \phi / \partial T$. Elsasser's final diagram is obtained by an equal area transformation, setting $x = -aT^2$ and $y = Q/2aT$, where a is an arbitrary constant.

Elsasser's chart was based chiefly on theoretical determinations of water vapour line intensities and frequencies. An alternative approach, based on laboratory and atmospheric radiation measurements, is due to Robinson (1947). Robinson also adopted a square-root pressure correction but assumed that ϕ was a function only of absorbing mass. He argued that the effects of T and T_b on absorption act in opposite directions, so that the emissivity of a layer (as a fraction of F_b) should vary only slightly with temperature. This compensation strictly applies only to the emitting layer adjacent to the reference level. Robinson's chart, the Kew Radiation Chart, was designed especially for application to surface radiative fluxes, so that the first layer will in general contribute more than 50% of the radiation reaching the ground, and the neglect of the temperature variability of the emissivity will be the chief temperature error. These considerations are illustrated by the data of Table IV; an appropriate value for the m of the first layer might be 0.1 - 0.5 gm/cm^2.

TABLE IV. EMISSIVITY, AS A PERCENTAGE OF F_b, FOR A PRESSURE OF ONE ATMOSPHERE.

$T = T_b$ (°K)	$m(gm/cm^2)$:	0.05	0.1	0.2	0.5	1.0
300		36.4	41.8	48.5	60.3	71.5
260		37.8	43.4	50.2	61.9	72.9
220		42.5	48.4	55.5	67.3	77.8

For layers above the first layer, the effects of temperature on both emission and transmission must be considered. Here the assumption $\phi = \phi(m)$, only, will give rise to more serious errors, but their effect on the total flux will be less than the assumption of an emissivity independent of temperature. With Robinson's assumptions (19) becomes, simply,

$$F = -\int_{m=0}^{m=m'} F_b \, d\phi(m).$$

This integral is clearly an area on the diagram with coordinates $(-\sigma T^4)$ and ϕ (m) (The Kew Radiation Chart). If tables of F_b (T) and ϕ(m) are available, the flux can be determined rapidly by a numerical integration, summing up $- F_b \Delta \phi$ (m) for a series of layers. Since Robinson used atmospheric radiation measurements for his ϕ values, and reduced his absorbing masses, determined from radiosonde flights, by a square-root pressure correction, it would be necessary to maintain this pressure treatment, even though incorrect, until a revised set of ϕ values, based on a linear pressure correction, is available.

Mention should be made here of a recent paper by Kaplan (1952), in which he outlines a technique for flux determination that adequately takes into account the effects of pressure and temperature. Kaplan adopts Robinson's (1947) empirical relation between intensity and flux, namely that the radiative flux from a layer m is approximately the same as the radiative intensity from a column of 1.66 m. Thus (6) becomes, for a single absorber,

$$F = -\int_0^\infty d\lambda \int_0^{m'} E_\lambda \, d\left[\exp\left\{-1.66\int_0^m k_\lambda \, dm\right\}\right]. \quad (21)$$

Kaplan now replaces dm, for water vapour, by qdp/g, where q is the specific humidity and g the acceleration due to gravity, and uses for k_λ a relation obtained by Elsasser (1942) on the assumption of equi-spaced lines of equal intensity, S. The bracketed function is found to depend on $\overline{qS}$, the mean value for the layer of the product of specific humidity and line intensity, and on the values of β (i.e. of $2\pi\alpha/d$) at the top and bottom of the layer, i.e. at p and p_0, where p_0 represents now the pressure at the reference level. In a spectral range for which S varies rapidly with temperature, it might be necessary to integrate from p_0 to p in several steps, since q also may vary rapidly in the vertical. The bracketed expression can now be integrated, i.e. averaged, over a single mean line, for which E_λ is essentially constant. If this latter integral is denoted by $\psi(\overline{qS}, \beta_0, \beta)$ and if $E_\lambda \Delta\lambda$ is replaced by $\Delta F_b/\pi$ (a function of the spectral region considered and of the temperature at level p), (21) can be written as:

$$F = -\pi^{-1} \sum \int_{p_0}^{p'} \Delta F_b \, d\psi(\overline{qS}, \beta_0, \beta). \quad (22)$$

In the general case the summation will have to be performed over a large number of spectral intervals, perhaps 50 - 100 for accuracy. This involves, however, the introduction of atmospheric sounding data into the problem prior to the summation over all wave lengths, and still leaves a numerical integration in the vertical to sum up the transmitted emission from all layers above the reference level. The chief advantage of Kaplan's technique is that the pressure and temperature effects are accurately incorporated into the evaluation, provided that ψ, a deep-layer intensity transmission, which must be determined to a number of levels above p_0, is obtained from a series of layers, for each of which $\overline{qS}$ is computed separately.

The technique suggested in the present paper involves no more labour, over the entire spectrum, than Kaplan's method for a single spectral interval. If the transmissions, ϕ, could be obtained accurately, the only source of error is the multiple-layer integrated flux transmission technique. Checks on such computations suggest that the maximum error in total radiative flux, from this cause, would be generally less than 0.1 percent. It will be some time before single-layer transmission data are available with better than this accuracy.

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